Exercise 1. Thanks to the Poincaré inequality, there exists a universal constant $C = C(\Omega)$ such that for all $u \in W_0^{1,2}(\Omega)$,

$$\|u\|_{\mathrm{L}^2(\Omega)} \leq C\,\|\nabla u\|_{\mathrm{L}^2(\Omega)}\,.$$

Since $f \in L^\infty(\Omega)$, we deduce by Cauchy-Schwarz inequality that

$$\left|\int_\Omega f(x)u(x)dx\right| \leq \|f\|_{\mathrm{L}^2(\Omega)}\,\|u\|_{\mathrm{L}^2(\Omega)} \leq C\,\|f\|_{\mathrm{L}^2(\Omega)}\,\|\nabla u\|_{\mathrm{L}^2(\Omega)}\,.$$

Therefore, we deduce that for all $u \in W_0^{1,2}(\Omega)$, we have

$$\begin{aligned} E(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega f\,u\,dx \geq \frac{1}{2}\int_\Omega |\nabla u|^2 dx - C\,\|f\|_{\mathrm{L}^2(\Omega)}\,\|\nabla u\|_{\mathrm{L}^2(\Omega)} \\ &\geq \frac{1}{4}\int_\Omega |\nabla u|^2 dx - C^2\int_\Omega f^2 dx. \end{aligned}$$

where we used the elementary inequality $ab \leq \frac{1}{4}a^2 + b^2$ $(a, b \in \mathbb{R})$. Therefore, if $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,2}(\Omega)$ is a minimising sequence, we deduce that

$$\frac{1}{4}\int_\Omega |\nabla u_n|^2 dx \leq E(u_n) + C^2\int_\Omega f^2 dx,$$

which shows by the Poincaré inequality that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,2}(\Omega)$. Therefore, up to a subsequence, there exists $u \in W^{1,2}(\Omega)$ such that $u_n \underset{n\to\infty}{\rightharpoonup} u$ and by the Rellich-Kondrachov theorem, we also have the strong convergence $u_n \underset{n\to\infty}{\longrightarrow} u$ in $L^2(\Omega)$. Then, the weak convergence implies that

$$\int_\Omega |\nabla u|^2 \leq \liminf_{n\to\infty}\int_\Omega |\nabla u_n|^2 dx.$$

On the other hand, the strong convergence in L^2 shows that

$$\left|\int_\Omega f\,u\,dx - \int_\Omega f\,u_n\,dx\right| \leq \|f\|_{\mathrm{L}^2(\Omega)}\,\|u_n - u\|_{\mathrm{L}^2(\Omega)} \underset{n\to\infty}{\longrightarrow} 0.$$

Since $E(u_n) \underset{n\to\infty}{\longrightarrow} m$, we deduce that

$$E(u) \leq \liminf_{n\to\infty} E(u_n) = m,$$

which shows that u is a minimiser of E on $W_0^{1,2}(\Omega)$. A standard computation as done in the lecture notes shows that u solves (in the distributional sense) the equation

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

We then say that u is the solution of the Dirichlet problem $\Delta u = f$ with Dirichlet boundary value. The unicity is trivial. If u_1 and u_2 are two solutions, the function $v = u_1 - u_2$ is harmonic ($\Delta v = 0$) and vanishes on the boundary, which shows by Stokes' formula that

$$\int_\Omega |\nabla v|^2 dx = \int_\Omega v\,\partial_\nu v\,d\mathscr{H}^{d-1} - \int_\Omega v\,\Delta v\,dx = 0.$$

Therefore, v is a constant function, but as v vanishes on the boundary, we deduce that $v = 0$.

Exercise 2. The proof is exactly similar and we omit it. We simply point out that two applications of the Poincaré inequality show that for all $u \in W_0^{2,2}(\Omega)$, we have

$$\int_\Omega u^2 dx \leq C \int_\Omega |\nabla u|^2 dx \leq C' \int_\Omega |\nabla^2 u|^2 dx.$$

Furthermore, we easily show that for all $u \in W_0^{2,2}(\Omega)$, we have

$$\int_\Omega (\Delta u)^2 dx = \int_\Omega |\nabla^2 u|^2 dx,$$

so the coercivity follows as before, and the rest of the proof is similar. To see that, for all $u \in C_c^\infty(\Omega)$ simply integrate by parts:

$$\begin{aligned}
\int_\Omega |\nabla^2 u|^2 dx &= \sum_{i,j=1}^d \int_\Omega \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 dx \\
&= -\sum_{i,j=1}^d \int_\Omega \left(\frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_j}\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) dx \\
&= -\sum_{i,j=1}^d \int_\Omega \left(\frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i}\left(\frac{\partial^2 u}{\partial x_j^2}\right) dx \\
&= \sum_{i,j=1}^d \int_\Omega \left(\frac{\partial^2 u}{\partial x_i^2}\right)\left(\frac{\partial^2 u}{\partial x_j^2}\right) dx \\
&= \int_\Omega \left(\sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}\right)\left(\sum_{j=1}^d \frac{\partial^2 u}{\partial x_j^2}\right) dx = \int_\Omega (\Delta u)^2 dx,
\end{aligned}$$

where we used the theorem of Schwarz for smooth functions. The general result follows by density. Finally, the Euler-Lagrange is given by

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases}$$

Integration by parts shows as above that if $v \in W_0^{2,2}(\Omega)$ is biharmonic ($\Delta^2 v = 0$), then $\Delta v = 0$ identically, and we can apply this result once more to deduce the uniqueness.

Exercise 3. 1. If $u \in W^{1,1}(I)$ and $u(1) = 1$ and $u(-1) = -1$, we deduce by the triangle inequality that

$$E(u) \geq \int_{-1}^1 |u'(x)| dx \geq \left|\int_{-1}^1 u'(x) dx\right| = |u(1) - u(-1)| = 2.$$

Therefore, we have $E(u) \geq 2$ for all $u \in W_g^{1,1}(I)$. Then, define for all $n \geq 1$

$$u_n(x) = \begin{cases} \operatorname{sgn}(x) & \text{for all } |x| > \dfrac{1}{n} \\ n\,x & \text{for all } -\dfrac{1}{n} \leq x \leq \dfrac{1}{n}. \end{cases}$$

Then $u_n \underset{n\to\infty}{\longrightarrow} \mathrm{sgn}$ in $L^1(I)$ and

$$E(u_n) = \int_{-\frac{1}{n}}^{\frac{1}{n}} n\,dx + \int_{-\frac{1}{n}}^{\frac{1}{n}} |u_n(x) - \mathrm{sgn}(x)|dx \underset{n\to\infty}{\longrightarrow} 2$$

as $u_n - \mathrm{sgn}$ is a bounded function.

2. $E(u) = 2$ if and only if

$$\int_{-1}^{1} |u(x) - \mathrm{sgn}(x)|dx = 0,$$

which shows that $u(x) = \mathrm{sgn}(x)$ for almost all $x \in]-1,1[$. However, this function does not admit a continuous representative, which shows that by the Sobolev embedding theoerem $W^{1,1}(I) \hookrightarrow C^0(I)$ that $u \notin W^{1,1}(I)$. In particular, E does not admit a minimiser in $W^{1,1}_g(I)$.

Exercise 4. 1. We can simply take $u_n(x) = \sin(nx)\sin(t)$.

2. The second part follows directly from the first chapter on the Euler-Lagrange equation: for all $\varphi \in C^\infty_c(\Omega)$, we have

$$\begin{aligned} E(u+\varphi) &= E(u) + \int_\Omega \left(\frac{\partial u}{\partial t}\cdot\frac{\partial \varphi}{\partial t} - \frac{\partial u}{\partial x}\cdot\frac{\partial \varphi}{\partial t}\right)dx\,dt + E(\varphi) \\ &= E(u) - \left[\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}\right](\varphi) + E(\varphi). \end{aligned}$$

Exercise 5. Take

$$\xi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \xi_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, we have

$$\frac{1}{2}f_1(\xi_1) + \frac{1}{2}f_1(\xi_2) = 0 < f_1\left(\frac{1}{2}\xi_1 + \frac{1}{2}\xi_2\right) = f_1\left(\frac{1}{2}\mathrm{Id}_2\right) = \frac{1}{16},$$

and likewise,

$$\frac{1}{2}f_2(\xi_1) + \frac{1}{2}f_2(\xi_2) = 1 < f_1\left(\frac{1}{2}\xi_1 + \frac{1}{2}\xi_2\right) = \frac{5}{4}.$$